Scalar waves on a naked-singularity background

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Abstract

We obtain global space-time weighted-L^2 (Morawetz) and L^4 (Strichartz) estimates for a massless chargeless spherically symmetric scalar field propagating on a super-extremal (overcharged) Reissner-Nordström background. To do this we first discuss the well-posedness of the Cauchy problem for scalar fields on non-globally hyperbolic manifolds, review the role played by the Friedrichs extension, and go over the construction of the function spaces involved. We then show how to transform this problem to one about the wave equation on the Minkowski space with a singular potential, and prove that the potential thus obtained satisfies the various conditions needed in order for the estimates to hold.

1 Introduction

There have been many studies of the well-posedness and decay of scalar fields in a given space-time whose metric satisfies the Einstein equations of general relativity, both as a precursor to the study of the stability of that metric, and as a means of probing various censorship conjectures. Although our main goal in this paper is to obtain estimates for scalar fields, we need to address the issue of well-posedness as well.

1.1 Well-posedness of the Cauchy problem for scalar fields

This question arises in studying the phenomenon of singularities in general relativity. The classical notion of a singularity of space-time is that of geodesic incompleteness. The weak cosmic censorship conjecture states that generically, singularities of spacetime must be hidden inside black holes, instead of being "naked", i.e. visible to distant observers. The strong form of this conjecture posits that generically, spacetimes must be globally hyperbolic, i.e. possess a complete spacelike hypersurface such that every causal curve in the manifold intersects it at exactly one point. Global hyperbolicity ensures that the spacetime has deterministic dynamics, since there is a Cauchy surface whose domain of dependence is the entire spacetime. In the case where the space-time is not globally hyperbolic, which can happen for example if the metric is singular[1] on a time-like curve, well-posedness of the Cauchy problem for the scalar

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[1]The minimum regularity required of the metric for the local existence and uniqueness of geodesics to hold is $C^{1,\alpha}$, $\alpha > 0$, so that any point in the manifold where the metric fails to be $C^{1,\alpha}$ needs to be removed, which can easily result in the loss of global hyperbolicity [6]. We note however, that the singularities of metrics considered in this paper are in fact much stronger, and the curve-integrability conditions of [6] do not hold for these examples.

wave equation has been suggested [24] as a substitute for geodesic completeness in determining how singular the spacetime actually is, and perhaps to see if quantum effects have any chance of regularizing the dynamics and restoring predictability [10]. This is particularly relevant if censorship conjectures somehow fail to be true and naked singularities turn out to be more abundant than otherwise allowed by them.

Let us now recall the set-up from [24]: Let $(\mathcal{M}, g)$ be a static, stably causal space-time, i.e., one that admits a hypersurface-orthogonal Killing vector field T^μ whose orbits are complete and everywhere timelike. A massless scalar field on $\mathcal{M}$ satisfies the equation

$$g^{\mu\nu}\nabla_\nu\nabla_\mu\psi = 0. \tag{1}$$

Suppose we specify initial data for ψ on a hypersurface Σ that is everywhere orthogonal to T^μ. If $\mathcal{M}$ is not globally hyperbolic, Σ will not be a Cauchy surface and data on Σ will determine ψ only on the domain of dependence $D(\Sigma)$. The aim of [24] was to define a physically sensible recipe for determining ψ everywhere in $\mathcal{M}$. To this end, one rewrites (1) in the form

$$\partial_t^2\psi = \alpha D^a(\alpha D_a\psi)$$

Here $\alpha = \sqrt{-g(T,T)}$, t is the Killing parameter, and D_a is the covariant derivative of the Riemannian metric induced on Σ. One then views

$$A := -\alpha D^a(\alpha D_a)$$

as an operator on the Hilbert space

$$\mathcal{H} = L^2(\Sigma, \alpha^{-1}d\sigma),$$

where $d\sigma$ denotes the induced volume form of Σ. The point of this definition is that with respect to the inner product of $\mathcal{H}$, A will now be symmetric and positive, although not yet self-adjoint since the initial domain of A would have to consist of sufficiently smooth functions. If we take this initial domain to be $C_c^\infty(\Sigma)$ (smooth functions of compact support on Σ) then A is also densely defined, and the equation we want to solve is

$$(\partial_t^2 + A)\psi = 0, \qquad \psi_{|\Sigma} = f, \qquad \partial_t\psi_{|\Sigma} = g. \tag{2}$$

It is then a classical result [20], that the above problem is well-posed, provided one replaces A in this equation with (one of) its self-adjoint extension(s) A_E. Such self-adjoint extensions are guaranteed to exist for real, symmetric operators [23]. The extension may not be unique though, which would imply that there is still an ambiguity about the dynamics. This may be interpreted as having to specify boundary conditions for the scalar field "on the singularity". One finds that there are three proposals for removing the ambiguity and restoring determinism to the dynamics of scalar fields:

(1) One can of course restrict attention only to the cases where the self-adjoint extension is unique (the so-called essentially self-adjoint case), and declare that spacetimes where the operator A is not essentially self-adjoint are "quantum-mechanically singular" [10] (meaning the singularity remains even if quantum particle dynamics, i.e waves, are considered in place of classical particle dynamics, i.e. geodesics). In this approach, the naked singularities present at the center in both the negative mass Schwarzschild ($m < 0$) and the super-extremal Reissner-Nordström ($|e| > m$) spacetimes are quantum-mechanically singular, and nothing more can be said about the evolution of scalar fields on these backgrounds.

(2) Another possibility is to characterize the singularity that leads to non-unique self-adjoint extensions for A as having a $U(N)$ "hair," [11] where N is the common value of the two deficiency indices $n_\pm := \dim(\ker(A^* \pm i))$ of A. This is because on the one hand there is a one-to-one correspondence between the self-adjoint extensions of A and unitary maps from $\ker(A^*+i)$ onto $\ker(A^*-i)$ [23], and on the other hand, as shown in [12], any "reasonable" way of defining the dynamics on the whole spacetime, would necessarily have to arise from *some* self-adjoint extension of A. We will show that the naked singularity of the super-extremal Reissner-Nordström solution is in this sense, quite hairy (see Remark 2).

(3) A third approach is to distinguish one self-adjoint extension from among all the possible ones, as being somehow more "natural" or "physical". It was suggested in [24] that the Friedrichs extension of A, i.e. the one coming from extending the corresponding quadratic form, is such a natural choice. We note that the Friedrichs extension of A always exists, and is unique by construction[2]. Moreover, as was shown in [21], the Friedrichs extension is the only self-adjoint extension of A under which the resulting dynamics of (2) agrees with that of the corresponding first-order (Hamiltonian) formulation,

$$\partial_t \Psi = -h\Psi,$$

with

$$\Psi = \begin{pmatrix} \psi \\ \alpha^{-1}\partial_t\psi \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & -\alpha \\ \alpha^{-1}A & 0 \end{pmatrix},$$

obtained via the (unique) skew-adjoint extension of the corresponding operator h.

In this paper we will take the third approach, and work with the Friedrichs extension of A. This is mainly because finiteness of the H^1 norm is used several times in the course of the proof of our estimates, and among the extensions of A, the Friedrichs extension is the only one whose domain is contained in H^1. It may be that some of our estimates hold for the other extensions of A as well, but our proof does not extend to those cases.

1.2 Estimates for scalar fields

The next natural question to consider for scalar waves after well-posedness is obtaining estimates for them. This is relevant among other things to the question of stability of the metric as a solution of the field equations, since expanding the perturbation functions in tensor harmonics yields a sequence of scalar wave equations [17], and having good estimates for each one of them seems necessary–although not sufficient–for proving linear stability of the metric under a small perturbation of its data. Moreover, the question of stability has an obvious connection to cosmic censorship if the solution to be perturbed happens to have a naked singularity.

Most of the results obtained so far in the direction of estimates concern either boundedness of the field [13, 25, 1] or its dispersion, i.e. the decay rate, with respect to the foliation parameter, of the supremum of the field on the slices of a given time-like or null foliation of a portion of the space-time [19, 2, 14, 15, 9, 7, 18, 16, 8]. In this paper we take a different approach to this problem, by obtaining estimates that bound the *global* space-time L^4-norm (and weighted-L^2 norm) of the field, in terms of appropriate norms of its data on a given Cauchy hypersurface (see [3, 4] for a similar approach). Such Strichartz (resp. Morawetz) estimates,

[2] The claimed non-uniqueness of the H^1-based extension (which is equivalent to the Friedrichs extension) of this operator in the case of superextremal Reissner-Nordström, [11, §IV.A.2], is due to the authors' error in not realizing the Sobolev space H^1 as the closure of C_c^∞ functions under the corresponding norm (see Section 3).

as they are known in the literature, are a staple of the modern theory of partial differential equations, and have proved very useful in the analysis of nonlinear evolution problems. We will be looking at the simplest non-trivial situation possible, namely, a spherically symmetric scalar field on a given spherically symmetric and static background space-time[3]. Furthermore, the background here is allowed to have a naked singularity, the main example we have in mind being the super-extremal (naked) Reissner-Nordström solution of the Einstein-Maxwell system. Our main result is

THEOREM 1 *Let $(\mathcal{M}, g)$ be the Reissner-Nordström space-time manifold, with mass m and charge e, such that*

$$|e| > 2m,$$

and let ψ be a massless, chargeless, spherically symmetric, scalar field on $\mathcal{M}$. Then there exists a constant $C > 0$ (which may depend on e and m) such that

$$\|\psi\|_{L^4(d\mu_g)} + \|\rho^{-1}\psi\|_{L^2(d\mu_g)} \leq C\mathbf{E}_{1/2}(\psi), \tag{3}$$

where ρ denotes the area-radius coordinate on $\mathcal{M}$, defined in Section 2 and $\mathbf{E}_{1/2}(\psi)$ is the conserved $\frac{1}{2}$-energy of the field, defined in Section 3.

The outline of this paper is as follows: In Section 2 we introduce the family of static spherically-symmetric Lorentzian manifolds, define a global system of coordinates on them, and write down the evolution equation satisfied by a scalar field on such a manifold. In Section 3 we introduce the notion of self-adjoint extension that is necessary to make that evolution problem well-posed, and define the function spaces that are going to be used. In Section 4 we state the various conditions that need to be satisfied by the metric coefficients of a manifold in the family we are considering, such that a scalar field on it will satisfy the estimate (3). This is then proved by transforming the problem to one about the flat wave equation with a potential, appealing to the result in [5], and transforming back to the original problem. The last section contains the proof of the fact that the metric coefficients of the super-extremal Reissner-Nordström solution satisfy the above mentioned conditions. This is accomplished by reformulating the problem in the language of real algebraic curves and appealing to the compactness result of [22].

2 Scalar fields on static spherically symmetric backgrounds

2.1 The space-time

Consider a four-dimensional connected spherically symmetric static space-time $(\mathcal{M}, g)$. More precisely on $\mathcal{M}$ we assume a time-like action of $\mathbb{R}$ and a space-like action of $SO(3, \mathbb{R})$ commuting with it. These actions should be without fixed points, except that at most one $\mathbb{R}$-orbit is allowed to be $SO(3, \mathbb{R})$-fixed, in which case it will be called the time axis. We restrict our attention to the cases where the $SO(3, \mathbb{R})$-orbits off the time axis are spheres.

[3] It is possible to extend our result to fields that are a finite sum of spherical harmonics, as in [9], but we do not pursue it here since this approach is not likely to yield an estimate for the general, nonsymmetric case.

2.2 The coordinate system

Let $T = T(p)$ denote the Killing vector generating the $\mathbb{R}$-action at $p \in \mathcal{M}$ and let $A = A(p)$ denote the area of the $SO(3,\mathbb{R})$-orbit passing through the point $p \in \mathcal{M}$. We define two $\mathbb{R} \times SO(3,\mathbb{R})$-invariant functions on $\mathcal{M}$ as follows:

$$\alpha := \sqrt{-g(T,T)}, \qquad \rho := \sqrt{A/4\pi}.$$

The quotient space $(\mathcal{Q}, \overline{g}) := (\mathcal{M}, g)/SO(3,\mathbb{R})$ is a two-dimensional Lorentzian manifold.[4] We can assign coordinates t, r on $\mathcal{Q}$ in such a way that

$$\overline{g}_{ab}\, dy^a\, dy^b = \alpha^2(r)\left(-dt^2 + dr^2\right).$$

The Killing field is clearly just ∂_t in these coordinates. We pull back t and r to $\mathcal{M}$. It is easy to see that we can assign angular coordinates $\Omega = (\theta, \phi)$ on $\mathcal{M}$ in such a way that the metric becomes

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = \alpha^2(r)\left(-dt^2 + dr^2\right) + \rho(r)^2(d\phi^2 + \sin^2\phi d\theta^2), \tag{4}$$

and the action of $SO(3,\mathbb{R})$ is the usual action on the unit sphere. These coordinates are unique up to translations in the t, r coordinates and rotations in the θ, ϕ coordinates. If there is a time axis we can then arrange that $r = 0$ there.

We note that in these coordinates the volume form of the spacetime is

$$d\mu_g = \alpha^2 \rho^2 d\Omega dr dt,$$

while the induced volume form on the Cauchy hypersurfaces $t = \text{const.}$ is

$$d\sigma = \alpha\rho^2 d\Omega dr.$$

Here

$$d\Omega = \sin\phi d\theta d\phi$$

denotes the volume form on the standard 2-sphere. We also note that there are at least two other volume forms on these hypersurfaces that are natural to consider, namely

$$d\sigma' = \alpha d\sigma = \alpha^2\rho^2 d\Omega dr \qquad \text{and} \qquad d\sigma'' = \alpha^{-1} d\sigma = \rho^2 d\Omega dr.$$

The significance of $d\sigma'$ is that L^p-norms based on it behave as one would expect with respect to the t-foliation: For $f : \mathcal{M} \to \mathbb{R}$ and all $p \geq 1$,

$$\|f\|_{L^p(d\mu_g)} = \|\ \|f\|_{L^p(d\sigma')}\|_{L^p(dt)}.$$

The significance of $d\sigma''$ will become clear in the next section.

[4] Possibly a manifold with boundary if there is a time axis.

2.3 The wave equation

Consider the action functional $\mathcal{A} = \frac{1}{2}\int_{\mathcal{M}} L d\mu_g$ corresponding to a real massless scalar field $\psi : \mathcal{M} \to \mathbb{R}$. The Lagrangian density is

$$L = g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi.$$

In the above-given coordinates

$$L = -\alpha^{-2}\psi_t^2 + \alpha^{-2}\psi_r^2 + \rho^{-2}|\not\nabla\psi|^2,$$

where $\not\nabla$ denotes the unit-spherical gradient, and

$$|\not\nabla\psi|^2 = |\partial_\phi\psi|^2 + \frac{1}{\sin^2\phi}|\partial_\theta\phi|^2.$$

A scalar field ψ that is a stationary point of the action $\mathcal{A}$, subject to a given set of initial values (ψ_0, ψ_1) on the Cauchy hypersurface $t = 0$, satisfies the following Cauchy problem

$$\partial_t^2\psi + A\psi = 0, \qquad \psi(0) = \psi_0, \qquad \partial_t\psi(0) = \psi_1, \tag{5}$$

where

$$A := -\frac{1}{\rho^2}\partial_r(\rho^2\partial_r) - \frac{\alpha^2}{\rho^2}\not\Delta, \tag{6}$$

with $\not\Delta$ denoting the Laplace-Beltrami operator on the unit 2-sphere. Note that the operator A is symmetric and positive definite with respect to the inner product given by the volume form $d\sigma''$, i.e. for $\phi, \psi \in C_c^\infty(\Sigma_t)$,

$$\int_{\Sigma_t} \phi A\psi \; d\sigma'' = \int_{\Sigma_t} \psi A\phi \; d\sigma'',$$

and

$$\int_{\Sigma_t} \phi A\phi \; d\sigma'' = \int_{\Sigma_t} \phi_r^2 + \frac{\alpha^2}{\rho^2}|\not\nabla\phi|^2 \; d\sigma'' \geq 0,$$

with equality only if $\phi \equiv 0$.

3 Self-Adjoint Extensions, Energy, and Sobolev Norms

As explained in Section 1, the evolution problem (5) is only meaningful if the operator A there is replaced by (one of) its self-adjoint extension(s) A_E. In the event that the self-adjoint extension is not unique, the one we choose to pick is the Friedrichs extension A_F, obtained by extending the quadratic form naturally associated with the operator A, namely

$$Q_A(\phi) := \int_{\Sigma_t} \phi_r^2 + \frac{\alpha^2}{\rho^2}|\not\nabla\phi|^2 \; d\sigma''. \tag{7}$$

Thus the Cauchy problem we are actually studying is the following

$$\partial_t^2\psi + A_F\psi = 0, \qquad \psi_{|\Sigma} = f, \qquad \partial_t\psi_{|\Sigma} = g. \tag{8}$$

Let $T_{\mu\nu}$ be the energy tensor of the field, defined as

$$T_{\mu\nu} = \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}L.$$

In particular, in the above given coordinates

$$T_{00} = \frac{1}{2}\psi_t^2 + \frac{1}{2}\psi_r^2 + \frac{\alpha^2}{2\rho^2}|\nabla\!\!\!/ \psi|^2.$$

Let X be a time-like Killing vector field for $\mathcal{M}$. Then the one-form P defined by $P(Y) := T(X,Y)$ is divergence-free, i.e. $*\,d*P = 0$ and thus $*P$ is a conserved current. In particular, let $X = \partial_t$, then $P_0 = T_{00}$ and $P_i = T_{0i}$, so that the only nonzero component of $*P$ is $(*P)_{123} = \rho^2 T_{00}$, and the conserved quantity is the *energy*

$$\mathbf{E}[\psi] := \int_{\Sigma_t} \rho^2 T_{00}\; dr d\Omega = \int_{\Sigma_t} T_{00}\; d\sigma''.$$

Thus using the $d\sigma''$ volume form, T_{00} is identified with the *energy density* of the field.

It therefore makes sense to also use $d\sigma''$ to define Sobolev spaces on the Σ_t's: Let $\mathbf{H}^1(\Sigma_t)$ denote the completion of smooth compactly supported functions on Σ_t with respect to the norm

$$\|f\|^2_{\mathbf{H}^1} := \int_{\Sigma_t} |f_r|^2 + \frac{\alpha^2}{\rho^2}|\nabla\!\!\!/ f|^2\; d\sigma''.$$

We define $\mathbf{H}^0(\Sigma_t)$ in the same way, i.e. completion with respect to the norm

$$\|f\|^2_{\mathbf{H}^0} := \int_{\Sigma_t} |f|^2\; d\sigma''.$$

The Sobolev spaces $\mathbf{H}^s(\Sigma_t)$ for $0 < s < 1$ are then defined via interpolation between the above two spaces, and one uses duality to define them also for $-1 \leq s < 0$. We note that by these definitions,

$$Q_A(\phi) = \|\phi\|^2_{\mathbf{H}^1(\Sigma_t)},$$

and moreover

$$\mathbf{E}[\psi] = \frac{1}{2}\left(\|\psi\|^2_{\mathbf{H}^1(\Sigma_t)} + \|\psi_t\|^2_{\mathbf{H}^0(\Sigma_t)}\right).$$

More generally, for $0 \leq s \leq 1$ we can define the s-energies

$$\mathbf{E}_s[\psi] := \frac{1}{2}\left(\|\psi\|^2_{\mathbf{H}^s(\Sigma_t)} + \|\psi_t\|^2_{\mathbf{H}^{s-1}(\Sigma_t)}\right),$$

and it is not hard to see that they are all conserved under the flow (8):

$$\frac{d}{dt}\mathbf{E}_s[\psi] = 0.$$

4 Transferring to Minkowski Space

It is clear that if we can view (5) as an evolution in Minkowski space, we may then be able to apply known theorems in order to get the estimates we want. This is of course only possible if the manifold $\mathcal{M}$ has at least the same topology as $\mathbb{R}^4$ (or $\mathbb{R}^4$ minus a line, in order to allow a singular time axis). We will be making this assumption from now on, and denote the coordinates on $\mathbb{R}^4$ by the same letters as those on $\mathcal{M}$, namely $(t, r, \Omega) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^2$, and a t-slice in $\mathbb{R}^4$ is still denoted by Σ_t. Let us define $u : \mathbb{R}^4 \to \mathbb{R}$ by

$$u(t, r, \Omega) := \frac{\rho(r)}{r}\psi(t, r, \Omega).$$

Note that

$$\|u\|^2_{L^2(\Sigma_t)} = \int_{\Sigma_t} |u|^2 r^2 dr d\Omega = \int_{\Sigma_t} |\psi|^2 d\sigma'' = \|\psi\|^2_{\mathbf{H}^0(\Sigma_t)}. \tag{9}$$

We can easily check that for $\psi \in C^\infty_c(\Sigma_t)$ we can perform an integration by parts to obtain

$$\begin{aligned} Q_A(\psi) &= \int_{\Sigma_t} \left[u_r^2 + \frac{\alpha^2}{\rho^2}|\not\nabla u|^2 + \frac{\rho^2}{r^2}[\partial_r(\frac{r}{\rho})]^2 u^2 + \frac{\rho}{r}\partial_r(\frac{r}{\rho})\partial_r(u^2)\right] r^2 dr\, d\Omega \\ &= \int_{\Sigma_t} \left[u_r^2 + \frac{\alpha^2}{\rho^2}|\not\nabla u|^2 + V(r)u^2\right] r^2 dr d\Omega =: Q_B(u), \end{aligned} \tag{10}$$

where the "potential" V is defined by

$$V(r) = \rho''(r)/\rho(r). \tag{11}$$

Let B denote the operator whose associated quadratic form is Q_B defined above, i.e.

$$B := -\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{\alpha^2}{\rho^2}\not\Delta + V.$$

We then have that

$$\partial_t^2 + A = \frac{r}{\rho}(\partial_t^2 + B)\frac{\rho}{r}.$$

Note that the first term in the definition of B coincides with the radial flat Laplacian. In particular, if ψ satisfying (5) is spherically symmetric, i.e. $\psi = \psi(t,r)$, then u can be viewed as a *radial* solution of the flat wave equation with a potential

$$u_{tt} - u_{rr} - \frac{2}{r}u_r + V(r)u = 0, \qquad u(0,r) = f(r), \qquad u_t(0,r) = g(r), \tag{12}$$

with $f = \rho\psi_0/r$ and $g = \rho\psi_1/r$.

Since by (10) the quadratic forms corresponding to A and B are equivalent, we can identify the Friedrichs extensions A_F and B_F of A and B, as well as the Sobolev spaces based on them, i.e. let $\mathcal{H}^s$ denote the B-based Sobolev space, defined to be the completion of smooth compactly supported functions on $\mathbb{R}^3 \setminus \{0\}$ with respect to the norm

$$\|f\|_{\mathcal{H}^s} := \|(B_F)^{s/2} f\|_{L^2}.$$

We then have that for $|s| \le 1$

$$\|u\|_{\mathcal{H}^s} = \|\psi\|_{\mathbf{H}^s}.$$

This is because the L^2 spaces agree by (9) while the H^1 spaces agree because of (10). On the other hand, if we let

$$P := -\Delta + V = -\frac{1}{r^2}\partial_r(r^2\partial_r) - \frac{1}{r^2}\not\Delta + V,$$

then it is clear that on the subspace of radial functions, B_F coincides with P_F (the Friedrichs extension of P), and so do the corresponding Sobolev spaces, which we will denote by $\mathcal{H}_{\text{rad}}$. In particular if we define the s-energy of u to be

$$\mathcal{E}_s[u] := \|u(t)\|_{\mathcal{H}^s_{\text{rad}}} + \|u_t(t)\|_{\mathcal{H}^{s-1}_{\text{rad}}},$$

then

$$\mathcal{E}_s[u] = \mathbf{E}_s[\psi] \tag{13}$$

and it is conserved by the flow (12).

In [5] it was shown that solutions to

$$(\partial_t^2 + P_F)u = 0 \tag{14}$$

satisfy certain weighted-L^2 (Morawetz) and L^p (Strichartz) spacetime estimates given below, provided the potential V meets certain criteria. In the case of a *radial* potential $V = V(r)$ on $\mathbb{R}^3$ these criteria reduce to the following three conditions on V:

$$\sup_{r>0} r^2 V(r) \quad < \quad \infty, \tag{15}$$

$$\inf_{r>0} r^2 V(r) \quad > \quad -1/4, \tag{16}$$

$$\sup_{r>0} r^2 \frac{d}{dr}(rV(r)) \quad < \quad 1/4. \tag{17}$$

The version of the main result in [5] for radial potentials (and general data) is as follows:

THEOREM 2 *Let $V \in C^1((0,\infty),\mathbb{R})$ satisfy (15,16,17), and let $P := -\Delta + V(|x|)$ where Δ is the Laplace operator with domain $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, and let P_F denote its Friedrichs extension. Then there exists a constant C, depending only on the quantities on the left in (15,16,17), such that any solution u of (14) satisfies*

$$\|r^{-1}u\|_{L^2(\mathbb{R}^4)} + \|u\|_{L^4(\mathbb{R}^4)} \leq C\mathcal{E}_{1/2}[u]. \tag{18}$$

Now, the space-time L^p norms of ψ and u are related in the following way:

$$\|\psi\|^p_{L^p(\mathcal{M})} = \int_{\mathcal{M}} |\psi|^p d\mu_g = \int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\int_0^\infty \alpha^2 \left(\frac{r}{\rho}\right)^{p-2} |u|^p \; r^2 dr\, d\Omega\, dt.$$

In particular

$$\|\psi\|^4_{L^4(\mathcal{M})} = \int_{\mathbb{R}^4} \left(\frac{\alpha r}{\rho}\right)^2 |u|^4 d^3x dt. \tag{19}$$

Similarly

$$\|\frac{\psi}{\rho}\|^2_{L^2(\mathcal{M})} = \int_{\mathbb{R}^4} \left(\frac{\alpha r}{\rho}\right)^2 \frac{|u|^2}{r^2} d^3x dt. \tag{20}$$

We thus have proved the following theorem regarding estimates for the scalar field ψ on $\mathcal{M}$:

THEOREM 3 *Let $\mathcal{M}$ be a Lorentzian manifold that is homeomorphic to $\mathbb{R}^4$, admitting a timelike $\mathbb{R}$ action and a spacelike $SO(3,\mathbb{R})$ action commuting with it, in such a way that exactly one $\mathbb{R}$-orbit, called Γ, is $SO(3,\mathbb{R})$-fixed. Let $(t,r,\Omega) \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{S}^2$ be the coordinate system on $\mathcal{M}$ as in Section 2, with $\Gamma = \{r = 0\}$. Let g be a Lorentzian metric on $\mathcal{M}$ that is of the form (4), is C^3 outside Γ, and is such that the functions ρ and α satisfy the following conditions*

(i) $\sup_{r>0}(r^2V) < \infty$

(ii) $\inf_{r>0}(r^2V) > -1/4$

(iii) $\sup_{r>0}(r^2 \frac{d}{dr}(rV)) < 1/4$

(iv) $\inf_{r>0}(\frac{\rho}{\alpha r}) > 0$

where

$$V(r) := \frac{1}{\rho}\frac{d^2\rho}{dr^2}.$$

Then there exists a constant $C > 0$, depending only on the quantities on the left in the conditions above, such that any spherically symmetric solution of

$$\partial_t^2\psi + A_F\psi = 0$$

satisfies

$$\|\rho^{-1}\psi\|_{L^2(\mathcal{M})} + \|\psi\|_{L^4(\mathcal{M})} \leq C\mathbf{E}_{1/2}[\psi]. \tag{21}$$

Proof: We simply observe that by (19,20),

$$\|\rho^{-1}\psi\|_{L^2} + \|\psi\|_{L^4} \leq \frac{1}{d}\|r^{-1}u\|_{L^2} + \frac{1}{\sqrt{d}}\|u\|_{L^4}$$

where d denotes the quantity on the left in condition (iv). The result then follows from (18) and (13).

■

5 Super-extremal Reissner-Nordström

The Reissner-Nordström manifold $(\mathcal{M}, g)$ is the static, spherically symmetric solution of Einstein-Maxwell equations. It is characterized by two parameters: mass m and charge e. It can be shown that for a metric of the form (4) to satisfy the Einstein-Maxwell system, one must have

$$\frac{d\rho}{dr} = \alpha^2 \tag{22}$$

and

$$\alpha = \sqrt{1 - \frac{2m}{\rho} + \frac{e^2}{\rho^2}}. \tag{23}$$

It thus follows that when $|e| < m$ the function ρ is bounded away from zero and there is no time axis. This is called the sub-extremal (black hole) case. When $|e| > m$ the manifold $\mathcal{M}$ has the same topology as $\mathbb{R}^4$ minus a line, the time axis is at $r = 0$ and the metric g is highly singular there. This is the super-extremal (naked) case of Reissner-Nordström.

The ODE (22) can be solved to express the “tortoise” coordinate r as a function of ρ. In the super-extremal case,

$$r(\rho) = r_0 + \rho + m\log(\frac{\rho^2 - 2m\rho + e^2}{e^2 - m^2}) + \frac{2m^2 - e^2}{\sqrt{e^2 - m^2}}\tan^{-1}\frac{\rho - m}{\sqrt{e^2 - m^2}}. \tag{24}$$

We choose the constant r_0 such that $r(0) = 0$. Since r is an increasing function of ρ this implicitly defines ρ as a function of r. It is also easy to compute from the ODE (22) that

$$\lim_{r\to 0}\frac{\rho}{r^{1/3}} = (3e^2)^{1/3}, \qquad \lim_{r\to\infty}\frac{\rho}{r} = 1.$$

and as a result

$$\lim_{r\to 0} r^{1/3}\alpha = (e/3)^{1/3}, \qquad \lim_{r\to\infty} \alpha = 1,$$

so that condition (iv) of Theorem 3 is clearly satisfied since

$$\frac{\rho}{\alpha r} \to \infty \text{ as } r \to 0, \qquad \frac{\rho}{\alpha r} \to 1 \text{ as } r \to \infty.$$

To prove Theorem 1, it thus remains to check that the function

$$V = \frac{1}{\rho}\frac{d^2\rho}{dr^2} = \frac{2m}{\rho^3} - \frac{2e^2 + 4m^2}{\rho^4} + \frac{6me^2}{\rho^5} - \frac{2e^4}{\rho^6},$$

satisfies conditions (i-iii) of Theorem 3. In the case of condition (iii) this does not appear to be an easy task, because of the transcendental relation (24) between r and ρ, and the dependence on the two additional variables e and m. To overcome these difficulties, first we exploit the inherent scaling in the problem to eliminate e (effectively setting it equal to one) and then by forgetting the relationship between r and ρ and treating them as independent quantities, we transform the problem into one about polynomials in three variables, which we then solve by utilizing a few basic tools from the theory of real algebraic curves.

Remark 1 The lower bound on the charge-to-mass ratio in Theorem 1 is not sharp. The actual ratio for which condition (iii) of Theorem 3 starts to be violated is less than 2 (It is not hard to see that this condition is violated for the ratios close to 1). We have not attempted to find the precise cut-off point, since it is not known whether condition (iii) is necessary for the estimate (21) to hold. We also note that the charge-to-mass ratios that occur in nature, such as in elementary particles, are in fact very large[5].

We perform the scaling first: Let

$$x := \frac{r}{|e|}, \qquad y := \frac{\rho}{|e|}, \qquad z := \frac{m}{|e|}, \qquad v := e^2 V$$

We then have

$$\alpha^2 = 1 - \frac{2z}{y} + \frac{1}{y^2}.$$

Let $y = \phi_z(x)$ denote the solution to the following initial value problem

$$\frac{dy}{dx} = \alpha^2(y,z), \quad y(0) = 0.$$

The inverse function of ϕ_z is in fact explicit:

$$\phi_z^{-1}(y) = y + \log(y^2 - 2yz + 1) + \frac{2z^2 - 1}{\sqrt{1-z^2}} \tan^{-1} \frac{y\sqrt{1-z^2}}{1 - yz}.$$

Note that since we are considering the super-extremal case, the parameter z ranges from 0 to 1.

[5]The charge e and mass m appearing in (23) are in geometric rationalized units, with $G = c = \hbar = 1$. The corresponding quantities in unrationalized electrostatic units e^* and m^* are related to e and m in the following way: $m = \frac{G}{c^2}m^*$ and $e = \frac{\sqrt{G}}{c^2}e^*$. It follows that $e/m \approx 10^{21}$ for an electron and $\approx 2 \times 10^{18}$ for a proton.

From the ODE it is clear that

$$\lim_{x\to 0}\frac{(\phi_z(x))^3}{x}=3, \qquad \lim_{x\to\infty}\frac{\phi_z(x)}{x}=1. \tag{25}$$

We have

$$v(y,z)=\frac{1}{y}\frac{d^2y}{dx^2}=\frac{2z}{y^3}-\frac{2+4z^2}{y^4}+\frac{6z}{y^5}-\frac{2}{y^6}=\frac{2}{y^6}(yz-1)(y^2-2yz+1).$$

From (25) it follows that

$$\lim_{x\to 0} x^2 v(\phi_z(x),z) = -\frac{2}{9}, \tag{26}$$

$$\lim_{x\to\infty} x^2 v(\phi_z(x),z) = 0, \tag{27}$$

and thus conditions (i) and (ii) of Theorem 3 are satisfied since $-\frac{2}{9}>-\frac{1}{4}$ and v is clearly negative and increasing for $y<1/z$, positive for $y>1/z$, and smooth away from $x=0$.

Remark 2 Note however that since $-\frac{2}{9}<\frac{3}{4}$, the operator

$$Q:=-\partial_x^2-\frac{2}{x}\partial_x+v(\phi_z(x),z)$$

with domain $C_c^\infty(\mathbb{R}^+)$ will have non-unique self-adjoint extensions, i.e. is limit-circle at zero (see [20, Appendix to X.1]). In this sense, with respect to *spherically symmetric* scalar waves, the naked singularity at $x=0$ of the super-extremal Reissner-Nordström solution has a $U(1)$ hair. Note also that the angular momentum contribution to the full Laplace-Beltrami operator of this manifold is

$$\frac{\alpha^2(\phi_z(x),z)}{(\phi_z(x))^2}\ell(\ell+1)$$

where ℓ is the spherical harmonic degree. Since this behaves like $x^{-4/3}$ near the origin, its addition to Q does not change anything as far as self-adjoint extensions are concerned. It thus follows that with respect to scalar waves in general, this naked singularity is "infinitely hairy," i.e. the corresponding unitary group is certainly infinite-dimensional[6].

Consider now the expression in condition (iii) of Theorem 3:

$$\begin{aligned} x^2\frac{d}{dx}(xv(\phi_z(x),z)) &= x^2v(\phi_z(x),z)+x^3\frac{\partial v}{\partial y}(\phi_z(x),z)\alpha^2(\phi_z(x),z) \\ &= w(x,\phi_z(x),z), \end{aligned}$$

where

$$\begin{aligned} w(x,y,z) \;:=\; & \left(-\tfrac{6z}{y^4}+\tfrac{8+28z^2}{y^5}-\tfrac{52z+32z^3}{y^6}+\tfrac{20+76z^2}{y^7}-\tfrac{54z}{y^8}+\tfrac{12}{y^9}\right)x^3 \\ & +\left(\tfrac{2z}{y^3}-\tfrac{2+4z^2}{y^4}+\tfrac{6z}{y^5}-\tfrac{2}{y^6}\right)x^2. \end{aligned}$$

Let us define the function $j_z:\mathbb{R}^+\to\mathbb{R}^+$ by

$$j_z(y)=w(\phi_z^{-1}(y),y,z). \tag{28}$$

[6]This provides us with another reason for choosing the Friedrichs extension instead.

A long, routine computation of Taylor series shows that near $y = 0$,

$$j_z(y) = \frac{2}{9} - \frac{4 - z^2}{270} y^2 + O(y^3), \tag{29}$$

$$j_z'(y) = -\frac{4 - z^2}{135} y + O(y^2), \tag{30}$$

and

$$j_z''(y) = -\frac{4 - z^2}{135} + O(y). \tag{31}$$

while, for large y,

$$j_z(y) = -4zy^{-3} - 28zy^{-4} \log y + O(y^{-4}) \tag{32}$$

and

$$j_z'(y) = 12zy^{-4} + 112zy^{-5} \log y + O(y^{-5}). \tag{33}$$

The implied constants in O-terms are locally uniform in z. Thus j_z has a local maximum at $y = 0$, for $0 \leq z < 1$. We wish to prove that, for $z \leq 1/2$, this is in fact the global maximum of j_z. Since it is easy to see that $j_z \to 0$ as $y \to \infty$, it suffices to show that the only local maximum is the one at $y = 0$. To this end we first show that

PROPOSITION 1 *j_z has no stationary points of inflection as long as $z \leq 1/2$.*

Proof: We compute

$$j_z'(y) = \frac{\partial w}{\partial y}(\phi_z^{-1}(y), y, z) + \frac{\partial w}{\partial x}(\phi_z^{-1}(y), y, z) \frac{1}{\alpha^2(y, z)} = q_1(\phi_z^{-1}(y), y, z),$$

where

$$\begin{aligned} q_1(x,y,z) \quad := \quad & \Big(\tfrac{24z}{y^5} - \tfrac{40+140\,z^2}{y^6} + \tfrac{312\,z+192\,z^3}{y^7} \\ & \quad - \tfrac{140+532\,z^2}{y^8} + \tfrac{432z}{y^9} - \tfrac{108}{y^{10}} \Big) x^3 \\ & + \Big(-\tfrac{24z}{y^4} + \tfrac{64\,z^2+32}{y^5} - \tfrac{120z}{y^6} + \tfrac{48}{y^7} \Big) x^2 \\ & + \Big(\tfrac{4z}{y^3} - \tfrac{4}{y^4} \Big) x, \end{aligned}$$

and similarly we compute

$$j_z''(y) = \frac{1}{\alpha^2(y, z)} q_2(\phi_z^{-1}(y), y, z),$$

where

$$\begin{aligned} q_2(x,y,z) \quad := \quad & \Big(-\tfrac{120z}{y^4} + \tfrac{240+1080\,z^2}{y^5} + \tfrac{-3024\,z^3-2784\,z}{y^6} \\ & \quad + \tfrac{1360+9464\,z^2+2688\,z^4}{y^7} + \tfrac{-8312\,z-9856\,z^3}{y^8} \\ & \quad + \tfrac{2200+12032\,z^2}{y^9} - \tfrac{6048z}{y^{10}} + \tfrac{1080}{y^{11}} \Big) x^3 \\ & + \Big(\tfrac{168z}{y^3} + \tfrac{-932\,z^2-280}{y^4} + \tfrac{2072\,z+1216\,z^3}{y^5} \\ & \quad + \tfrac{-3356\,z^2-916}{y^6} + \tfrac{2688z}{y^7} - \tfrac{660}{y^8} \Big) x^2 \\ & + \Big(-\tfrac{60z}{y^2} + \tfrac{80+152\,z^2}{y^3} - \tfrac{284z}{y^4} + \tfrac{112}{y^5} \Big) x \\ & + \tfrac{4z}{y} - \tfrac{4}{y^2}. \end{aligned}$$

To prove the statement of the proposition, it is enough to show that the zero-sets of $q_1(x,y,z)$ and $q_2(x,y,z)$ are disjoint. The strategy is to forget about the transcendental relationship between x and y and treat them as independent variables, in order to be able to use results from the theory of real algebraic curves.

At a stationary point of inflection both j_z' and j_z'' would vanish and hence so would the resultant[7] of q_1, q_2, considering both as cubic polynomials in x. Up to an integer multiple, this resultant is computed[8] to be

$$R(q_1, q_2; x) = y^{-34}(yz-1)^2 p(y,z),$$

where

$$\begin{aligned}
p(y,z) \quad = \quad & \left(-1536\, y^9 + 4608\, y^7\right) z^9 \\
& + \left(3452\, y^{10} + 2040\, y^8 - 20100\, y^6\right) z^8 \\
& + \left(-3504\, y^{11} - 16456\, y^9 - 8608\, y^7 + 36120\, y^5\right) z^7 \\
& + \left(1947\, y^{12} + 20360\, y^{10} + 62966\, y^8 + 48272\, y^6 - 33769\, y^4\right) z^6 \\
& + \left(-576\, y^{13} - 11988\, y^{11} - 71800\, y^9 - 153832\, y^7 \right.\\
& \quad \left. - 104440\, y^5 + 17900\, y^3\right) z^5 \\
& + \left(72\, y^{14} + 3552\, y^{12} + 38762\, y^{10} + 143492\, y^8 \right.\\
& \quad \left. + 208760\, y^6 + 109100\, y^4 - 5530\, y^2\right) z^4 \\
& + \left(-432\, y^{13} + 10464\, y^{11} - 66316\, y^9 - 154672\, y^7 \right.\\
& \quad \left. - 151552\, y^5 - 62848\, y^3 + 972\, y\right) z^3 \\
& + \left(1152\, y^{12} + 15384\, y^{10} + 57803\, y^8 + 83120\, y^6 \right.\\
& \quad \left. + 58958\, y^4 + 20912\, y^2 - 81\right) z^2 \\
& + \left(-1440\, y^{11} - 10536\, y^9 - 21648\, y^7 - 20824\, y^5 \right.\\
& \quad \left. - 11680\, y^3 - 3888\, y\right) z \\
& + 720\, y^{10} + 2160\, y^8 + 2760\, y^6 + 1908\, y^4 + 944\, y^2 + 324.
\end{aligned} \tag{34}$$

Note that

$$q_1(x, 1/z, z) = -8x^2(z^2-1)z^5\left(2xz^3 - 2xz + 1\right),$$

so it is zero only for $x = 0$ or $x = \frac{1}{2z-2z^3}$. However,

$$q_2(1/(2z-2z^3), 1/z, z) = \frac{\left(5\,z^2-3\right) z^2}{1-z^2}.$$

$z = 0$ would correspond to $x = y = \infty$, which is not of interest, and $\sqrt{3/5} > \frac{1}{2}$. It is thus enough to show that $p(y,z)$ has no real zeros for $0 \le z \le \frac{1}{2}$. This is easy to establish for $z = 0$, since

$$p(y,0) = 720\, y^{10} + 2160\, y^8 + 2760\, y^6 + 1908\, y^4 + 944\, y^2 + 324 > 0 \tag{35}$$

[7] By definition, the resultant of two polynomials

$$q_1 = a_n \Pi_{i=1}^n (x - \alpha_i), \qquad q_2 = b_m \Pi_{i=1}^m (x - \beta_i)$$

is

$$R(q_1, q_2; x) = a_n^m b_m^n \Pi_{i=1}^n \Pi_{j=1}^m (\alpha_i - \beta_j)$$

It can be computed from the Euclidean algorithm, or as the determinant of Sylvester's matrix or Bezout's matrix.

[8] The necessary computations here and elsewhere in the paper are performed by the computer algebra package PARI using exact integer arithmetic.

for all y. We postpone the rest of the proof of this proposition until the end of the section, and instead show first how this implies the desired result about V, i.e. that it satisfies condition (iii) of Theorem 3.

LEMMA 5.1 *The function j_0 has no critical points other than at $y = 0$.*

Proof: We have for $z = 0$

$$\phi_0^{-1}(y) = y - \tan^{-1} y \tag{36}$$

and

$$q_1(x, y, 0) = -4y^{-10}x[(10y^4 + 35y^2 + 27)x^2 - 4(3y^3 + 2y^5)x + y^6].$$

Consider the algebraic curve $q_1(x, y, 0) = 0$. It has a branch $x = 0$. The other two branches can be transformed into a hyperbola by changing variables to

$$\xi := \frac{x}{y^3}, \qquad \eta := \frac{x}{y}.$$

Let $\mathfrak{H}$ denote the piece of this hyperbola that lies in the first quadrant of the (ξ, η) plane, i.e.

$$\mathfrak{H} = \{(\xi, \eta) \mid \xi \geq 0, \eta \geq 0, h(\xi, \eta) = 0\},$$

with

$$h := 10\eta^2 + 35\xi\eta + 27\xi^2 - 12\xi - 8\eta + 1. \tag{37}$$

Likewise, let $\mathfrak{T}$ denote the part of the transcendental curve $x = \phi_0^{-1}(y)$ in the same quadrant:

$$\mathfrak{T} = \{(\xi, \eta) \mid \xi \geq 0, \eta \geq 0, \tau(\xi, \eta) = 0\},$$

where τ has the following expansion

$$\tau = -1 + \frac{1}{3\xi} - \frac{1}{5\xi^2}\eta + \frac{1}{7\xi^3}\eta^2 - \ldots$$

Recall that we have already shown by (35) that the function j_0 cannot have a stationary point of inflection. Therefore on each branch of the hyperbola $\mathfrak{H}$ the second derivative j_0'' must be of one sign. Thus one branch must correspond to minima and the other to maxima of j_0.

It is not hard to see that $\mathfrak{H}_L$, the lower branch of the hyperbola $\mathfrak{H}$, is well separated from $\mathfrak{T}$: Let R be the rectangle $[0, 1/9] \times [0, 6/37]$ in the (ξ, η) plane. Then

Claim 1 *$\mathfrak{H}_L$ is contained in R while $\mathfrak{T} \cap R = \emptyset$.*

Proof: The ξ-intercept of $\mathfrak{H}_L$ is at $\xi = 1/9$. The tangent line to $\mathfrak{H}_L$ at the point $(1/9, 0)$ is $\eta = -\frac{54}{37}(\xi - \frac{1}{9})$. Since $\mathfrak{H}_L$ is concave down, it lies below this tangent, and the η-intercept of the tangent line is at $\eta = 6/37$ which shows the first part of the claim. Consider next the ODE satisfied by the curve $x = y - \tan^{-1} y$, i.e. $\frac{dx}{dy} = \frac{y^2}{y^2+1}$. For $y \leq 1$ we have $\frac{dx}{dy} \geq \frac{y^2}{2}$ which upon integration yields

$$x(y) \geq \frac{y^3}{6}, \tag{38}$$

and thus $\xi \geq \frac{1}{6}$, while if $y \geq 1$ then $\frac{dx}{dy} \geq \frac{1}{2}$ and thus, using the bound (38) at $y = 1$,

$$x(y) \geq \frac{1}{2}(y - 1) + x(1) \geq \frac{1}{2}(\frac{1}{3}y + \frac{2}{3}y) - \frac{1}{3} \geq \frac{1}{6}y,$$

and thus $\eta \geq \frac{1}{6}$. Thus along $\mathfrak{T}$ we have $\min(\xi, \eta) \geq 1/6$, which proves the second part of the claim.

Let $\mathfrak{H}_U$ denote the upper branch of the hyperbola $\mathfrak{H}$. It intersects $\mathfrak{T}$ at $\xi = \frac{1}{3}, \eta = 0$. This point corresponds to the origin of the (x, y) plane where we know that j_z has a local maximum. We have thus shown in the above Claim that j_0' cannot have any local minima. Moreover, we can compute the tangent lines to the two curves at $(1/3, 0)$ and we obtain that

$$\left(\frac{d\eta}{d\xi}\right)_{\mathfrak{H}_U} = -\frac{18}{11}, \qquad \left(\frac{d\eta}{d\xi}\right)_{\mathfrak{T}} = -\frac{5}{3}.$$

Thus $\mathfrak{H}_U$ is below $\mathfrak{T}$ near $(\frac{1}{3}, 0)$. It is easy to see that this is also the case near the η axis (which corresponds to the infinity of the (x, y) plane). Thus if $\mathfrak{T}$ were to intersect $\mathfrak{H}_U$ it would have to do so at least twice. Since a continuous function cannot have two consecutive local maxima without a minimum in between this is not possible and the curves $\mathfrak{H}_U$ and $\mathfrak{T}$ have no other common point in (the first quadrant of) the (ξ, η) plane. This proves the lemma.

■

The following Proposition completes the proof of Theorem 1 by showing that the quantity on the left in condition (iii) of Theorem 3 is equal to $\frac{2}{9}$.

PROPOSITION 2 *The function j_z for $0 \leq z \leq 1/2$ has no local maximum other than at $y = 0$.*

Proof: Let

$$K := \{(y, z) | y > 0,\ j_z'(y) = 0,\ j_z''(y) \leq 0\}.$$

Note that K is closed. From (30) we see that there is an $\epsilon > 0$ such that there $j_z' \neq 0$ in the region $0 < y < \epsilon$, $0 \leq z \leq \frac{1}{2}$. Similarly, from (33) we see that there is a Y such that $j_z' \neq 0$ in the region $Y < y < \infty$, $0 \leq z \leq \frac{1}{2}$. K is therefore compact, since it is a closed subset of $[\epsilon, Y] \times [0, \frac{1}{2}]$.

If K is nonempty, then there exists a point $(\bar{y}, \bar{z})$ in K with smallest z, i.e. $z \geq \bar{z}$ for all $(y, z) \in K$. Thus $j_{\bar{z}}'(\bar{y}) = 0$ and $j_{\bar{z}}''(\bar{y}) \leq 0$. By the previous lemma, $\bar{z} > 0$. Suppose that $j_{\bar{z}}''(\bar{y}) \neq 0$. Then by the implicit function theorem, we can solve

$$q_1(\phi_z^{-1}(y), y, z) = 0$$

near $(\bar{y}, \bar{z})$, to get the curve $y = \vartheta(z)$. Now for $z < \bar{z}$, $(\vartheta(z), z) \notin K$ so we have $j_z''(\vartheta(z)) > 0$. Thus by continuity, $j_{\bar{z}}''(\bar{y}) \geq 0$, which leads to a contradiction. Thus we must have $j_{\bar{z}}''(\bar{y}) = 0$. But this is ruled out by Proposition 1. Therefore K must be empty. This proves the statement of Proposition 2.

■

We conclude with the remainder of the proof of Proposition 1. First we need some definitions: For

$$p(y, z) = \sum_{k,l} c_{k,l} y^k z^l$$

a polynomial in two variables with real coefficients, let C be the curve

$$C := \{(y, z) \in \mathbb{R}^2 \mid p(y, z) = 0\}$$

considered as a subset of $\mathbb{R}^2$ in the usual topology. The *Newton Polygon* of p is defined to be the convex hull of the set

$$N := \{(k,l) \in \mathbb{Z}^2 \mid c_{k,l} \neq 0\}.$$

If E is an (oriented) edge of the Newton polygon with endpoints (k'_E, l'_E) and (k''_E, l''_E), then the numbers d_E, p_E and q_E are defined by

$$d_E := \gcd(k''_E - k'_E, l''_E - l'_E), \quad p_E := \frac{k''_E - k'_E}{d_E}, \quad q_E := \frac{l''_E - l'_E}{d_E},$$

and the *edge polynomial* $e_E \in \mathbb{R}[t]$ by

$$e_E(t) := \sum_{i=0}^{d_E} c_{k'_E + ip_E, l'_E + iq_E} t^i.$$

An edge is called *outer* if it maximizes some linear function $ak + bl$ on the Newton polygon, where at least one of a or b is positive.

The following compactness criteria for plane algebraic curves is proved in [22]:

THEOREM 4 *For the compactness of C it suffices that p is not divisible by y or z, and the edge polynomials corresponding to outer edges have no real zeros.*

The Newton polygon of the polynomial (34) is shown in Figure 1. It is obvious that p does

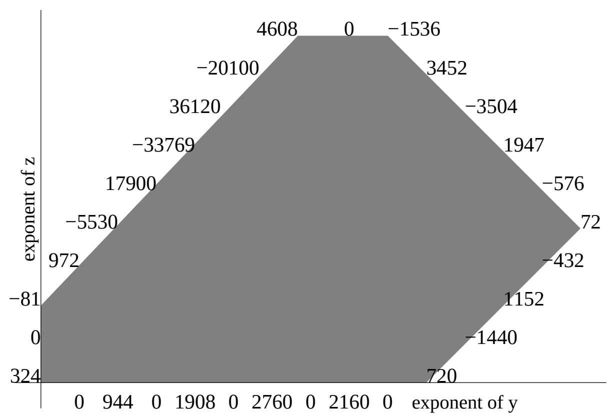

Figure 1: Newton polygon of p

not satisfy the criteria in this theorem, since some of the outer edge polynomials are of odd degree. To remedy this, we take a simple projective transformation of the plane and consider its composition with p: Let

$$q(z', y') := z'^{18} p(\frac{y'}{z'}, \frac{1}{z'}). \tag{39}$$

We need to show that q has no zeros in the region $[2, \infty) \times (0, \infty)$. The Newton polygon of q is the image of the Newton polygon of p under the linear map $(k,l) \mapsto (18 - k - l, k)$. It is shown in Figure 2.

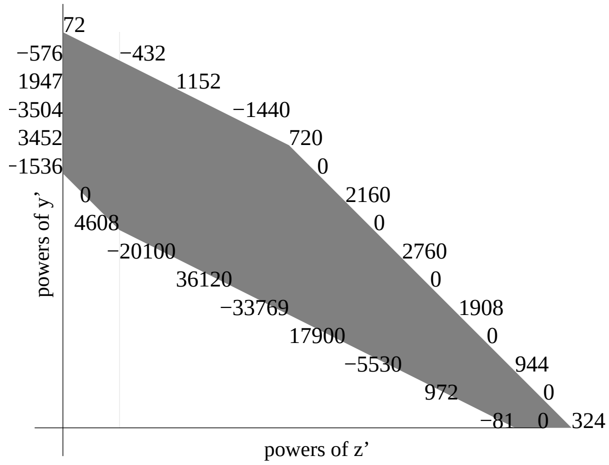

Figure 2: The Newton Polygon of q

There are two outer edges, with corresponding edge polynomials

$$\begin{aligned} e_1(t) &= 324t^{10} + 994t^8 + 1908t^6 + 2760t^4 + 2160t^2 + 720, & (40)\\ e_2(t) &= 72t^4 - 432t^3 + 1152t^2 - 1440t + 720. & (41) \end{aligned}$$

e_1 is clearly never zero, and the fact that e_2 has no real zeros is easily checked by Sturm's criterion. By the above Theorem, the zeroes of q are a compact set. If z' is maximal for this set then $\partial q/\partial y'$ is zero. The resultant of q and $\partial q/\partial y'$ with respect to y' must then be zero as well. This resultant is

$$R(q, \frac{\partial q}{\partial y'}; y') = z'^{114}(z'-1)^{45}(z'+1)^{45}(2z'-1)(2z'+1)f(z')^3 g(z')$$

where

$$\begin{aligned} f(z') = \; & 15268608z'^{12} - 91375200z'^{10} + 235796896z'^8 - 336360313z'^6 \\ & + 278925810z'^4 - 126777721z'^2 + 24542656 \end{aligned}$$

and

$$\begin{aligned} & g(z') = \\ & 2238642500162400000000z'^{32} - 34444148848863120000000z'^{30} \\ & + 237592851413120362800000z'^{28} - 985391370893335206960000z'^{26} \\ & + 2763859141396512532788000z'^{24} - 5568163844214174878032000z'^{22} \\ & + 8328390177670642537641736z'^{20} - 9407814473561334395291936z'^{18} \\ & + 8074051365793494550609047z'^{16} - 5249226125431353947046641z'^{14} \\ & + 2557646890465822492261299z'^{12} - 918262815118642547577717z'^{10} \\ & + 238729228492213678314693z'^8 - 44988866247608231183315z'^6 \\ & + 6473535589377087487753z'^4 - 739775558130688228967z'^2 \\ & + 49586421501845352448. \end{aligned}$$

f has no real zeros at all, while all the zeros of g are contained in $|z'| < 2$. Both of these facts are established by Sturm's criterion. It follows that the maximal value of z' is less that 2, and this concludes the proof of Proposition 1.

■

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References

[1] Horst R. Beyer. On the stability of the Kerr metric. *Comm. Math. Phys.*, 221(3):659–676, 2001.

[2] J. Bičak. Gravitational collapse with charge and small asymmetries, I. Scalar perturbations. *Gen. Rel. Grav.* 3:331-349, 1972.

[3] P. Blue and A. Soffer. Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates. *Adv. Differential Equations*, 8(5):595–614, 2003.

[4] P. Blue and A Soffer. The wave equation on the Schwarzschild metric II: Local decay for the spin 2 Regge-Wheeler equation. arXiv:gr-qc/0310066.

[5] Nicolas Burq, Fabrice Planchon, John Stalker, and A. Shadi Tahvildar-Zadeh. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay. arXiv:math.AP/0401019.

[6] C. J. S. Clarke. Generalized hyperbolicity in singular spacetimes. *Class. Quantum Grav.*, 15:975–984, 1998.

[7] V. Cardoso, S. Yoshida, O. J. C. Dias and J. P. S. Lemos, Late-Time Tails of Wave Propagation in Higher Dimensional Spacetimes. *Phys. Rev. D* 68 (2003) 061503 (Rapid Communications); arXiv:hep-th/0307122.

[8] Mihalis Dafermos and Igor Rodnianski. A proof of Price s law for the collapse of a self-gravitating scalar field. arXiv:gr-qc/0309115.

[9] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau. Decay rates and probability estimates for massive Dirac particles in the Kerr-Newman black hole geometry. *Comm. Math. Phys.*, 230(2):201–244, 2002.

[10] Gary T. Horowitz and Donald Marolf. Quantum probes of spacetime singularities. *Phys. Rev. D (3)*, 52(10):5670–5675, 1995.

[11] Akihiro Ishibashi and Akio Hosoya. Who's afraid of naked singularities? Probing timelike singularities with finite energy waves. *Phys. Rev. D (3)*, 60(10):104028, 12, 1999.

[12] Akihiro Ishibashi and Robert M. Wald. Dynamics in non-globally-hyperbolic static spacetimes. II. General analysis of prescriptions for dynamics. *Classical Quantum Gravity*, 20(16):3815–3826, 2003.

[13] Bernard S. Kay and Robert M. Wald. Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. *Classical Quantum Gravity*, 4(4):893–898, 1987.

[14] Hiroko Koyama and Akira Tomimatsu. Asymptotic tails of massive scalar fields in a Reissner-Nordström background. *Phys. Rev. D (3)*, 63(6):064032, 9, 2001.

[15] Hiroko Koyama and Akira Tomimatsu. Asymptotic tails of massive scalar fields in a Schwarzschild background. *Phys. Rev. D (3)*, 64(4):044014, 8, 2001.

[16] Matei Machedon and John G. Stalker. Decay of solutions to the wave equation on a spherically symmetric static background. *in preparation*.

[17] Vincent Moncrief. Odd-parity stability of a Reissner-Nordström black hole. *Phys. Rev. D (3)*, 9:2707, 1974.

[18] Eric Poisson. Radiative falloff of a scalar field in a weakly curved spacetime without symmetries. *Phys. Rev. D (3)*, 66(4):044008, 17, 2002.

[19] Richard Price. Nonspherical perturbations of relativistic gravitational collapse, I-II. *Phys. Rev. D (3)*, 5:2419–2438,2439–2454, 1972.

[20] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[21] Itai Seggev. Dynamics in stationary, non-globally hyperbolic spacetimes. arXiv:gr-qc/0310016.

[22] John G. Stalker. A compactness criterion for real plane algebraic curves. *preprint*, 2003.

[23] John Von Neumann. Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. *Math. Ann.*, 102:49–131, 1929-1930.

[24] Robert M. Wald. Dynamics in nonglobally hyperbolic, static space-times. *J. Math. Phys.*, 21(12):2802–2805, 1980.

[25] Bernard F. Whiting. Mode stability of the Kerr black hole. *J. Math. Phys.*, 30(6):1301–1305, 1989.

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